A theoretical study of slow steady flow of a viscous incompressible fluid between two porous walls with suction

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ABSTRACT: This study focusses on the solution of Navier-Stokes equations for the following condition: A slow steady motion of a viscous incompressible fluid between two porous walls at slightly variable distance from each other. At the two walls, equal suction velocities ±$v_0(x)$ have been imposed. The Fourier transform method has been used to attain the solution with the assumption that the distance between the walls has a small variation. Results have also been obtained for the particular case of sinusoidal variation.

Keywords: Suction, Perturbation, Sinusoidal variation, Unsteady, Porous walls, Oscillatory flow, Stratified fluid

INTRODUCTION

Ravi Kant (1980) considered the free and forced convection in a parallel channel formed by two conducting parallel porous walls. The unsteady flow of an incompressible viscous fluid between two parallel porous plates with injection at the lower plate and suction at the upper plate was studied by (Ganesh and Krishnambal, 2007). Effects of magnetic field and radiation on MHD fluid flow in a channel were studied by (Manjulatha et al., 2013). (Ganesh, 2014) considered the viscous oscillatory flow of a stratified fluid in a long vertical narrow rectangular channel. The flow of an incompressible viscous fluid between two parallel porous plates has been investigated by (Hafeez and Chifu, 2014). In the present study the solution of the problem of slow steady motion of a viscous incompressible fluid between two walls has been obtained when equal suction velocities ±$v_0(x)$ have been applied at the walls.

Equations of Motion and their Solution

We consider a two dimensional incompressible fluid flow between two porous walls at slightly variable distance. The origin has been taken midway between the walls, the $x$-axis and the $y$-axis vertically upwards. Navier-Stokes equations of motion for slow steady flow are

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu} \frac{\partial p}{\partial x}, \quad (1)$$
$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{1}{\mu} \frac{\partial p}{\partial y}, \quad (2)$$

and the equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (3)$$

where $u$, $v$ are components of velocity in the $x$ and $y$ directions and $p$ is the pressure.

Let the equations of the walls be

$$y = \pm(h+\varepsilon N(x)),$$

where $h$ is the constant and $\varepsilon N(x) << h$. Thus the boundary conditions are

$$u = 0, v = \pm v_0(x) \ \text{when} \ y = \pm(h+\varepsilon N(x)) \quad (4)$$

Let

$$u(x,y) = u_0 + u'(x,y)$$
$$u(x,y) = v'(x,y)$$
$$p(x,y) = p_0 + p'(x,y),$$

(5)
where primed equations denote perturbations due to variations in distance and \( u_0 \) and \( p_0 \) are known quantities in the unperturbed flow between two smooth parallel walls (\( \varepsilon = 0 \)). Now, \( p_0 \) is a function of \( x \) and \( dp_0/dx = \text{constant} = P \).

Hence
\[
p_0 = Px + c . \tag{6}
\]

Also, \( u_0 \) satisfies the differential equation
\[
\frac{\partial^2 u_0}{\partial y^2} = \frac{1}{\mu} \frac{dp_0}{dx}
\]
and as given by Pai, (1956)
\[
u_0 = \frac{P}{2\mu}(y^2 - h^2) . \tag{7}
\]

Substituting from (5) in the equations (1), (2) and (3), we obtain
\[
\frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2} = \frac{1}{\mu} \frac{dp'}{dx} \tag{8}
\]
\[
\frac{\partial^2 v'}{\partial x^2} + \frac{\partial^2 v'}{\partial y^2} = \frac{1}{\mu} \frac{dp'}{dy} \tag{9}
\]
\[
\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0 \tag{10}
\]
subject to the conditions
\[
u' = -u_0, v' = \pm v_0 \text{ when } y = \pm(h + \varepsilon N(x)) . \tag{11}
\]

Now, we introduce complex Fourier transform as
\[
U = \int_{-\infty}^{\infty} u' e^{-i\xi x} \, dx
\]
\[
V = \int_{-\infty}^{\infty} v' e^{-i\xi x} \, dx
\]
\[
F = \int_{-\infty}^{\infty} p' e^{-i\xi x} \, dx .
\]

Multiplying equations (8), (9) and (10) by \( e^{-i\xi x} \) and integrating with respect to \( x \) between \( -\infty \) to \( \infty \), we get
\[
\left( \frac{d^2}{dy^2} - \xi^2 \right) U = \frac{i\xi}{\mu} F \tag{12}
\]
\[
\left( \frac{d^2}{dy^2} - \xi^2 \right) V = \frac{1}{\mu} \frac{dF}{dy} \tag{13}
\]
\[
i\xi U + \frac{dV}{dy} = 0 . \tag{14}
\]

Eliminating \( U \) and \( F \) from the above equations, we obtain
\[
\left( \frac{d^2}{dy^2} - \xi^2 \right)^2 V = 0 . \tag{15}
\]

The general solution of (15) may be taken to be
\[
V = A\xi y \cosh(\xi y) + B \sinh(\xi y) . \tag{16}
\]

where \( A \) and \( B \) are independent of \( y \).

By applying Fourier inversion formula, we get
\[
v' = \frac{1}{2\pi} \int_{-\infty}^{\infty} [A\xi y \cosh(\xi y) + B \sinh(\xi y)] e^{i\xi x} \, d\xi . \tag{17}
\]

From (14)
\[
U = -\frac{1}{i\xi} \frac{dV}{dy} = -\frac{1}{i} \left[ (A + B) \cosh(\xi y) + A\xi y \sinh(\xi y) \right] . \tag{18}
\]

Thus, applying the inversion formula, we get
\[
u' = \frac{1}{2\pi} \int_{-\infty}^{\infty} [(A + B) \cosh(\xi y) + A\xi y \sinh(\xi y)] e^{i\xi x} \, d\xi . \tag{19}
\]

From (12),
By Fourier Integral theorem, if we write

\[ u' = -u_0, \quad v' = \pm v_0 \quad \text{when} \quad y = \pm (h + \varepsilon N(x)) \]

Making use of these conditions in (17) and (19), we get

\[ \frac{\partial^2 U}{\partial y^2} - \xi^2 U = \frac{\mu}{\nu} F \]

Substituting from (18) for \( U \) in this equation, we get

\[ p' = \frac{\mu}{\pi} \int_{-\infty}^{\infty} A \cosh(\gamma \xi) e^{i \xi \gamma} d \xi \]

The functions \( A \) and \( B \) are to be determined so as to satisfy the boundary conditions

\[ u' = -u_0, v' = \pm v_0 \quad \text{when} \quad y = \pm (h + \varepsilon N(x)) \]

Substituting these expressions in (21) and (22) and equating like powers of \( \varepsilon \), we obtain (on equating terms independent of \( \varepsilon \))

\[ v_0(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [A_0 \cosh(\xi h) + B_0 \sinh(\xi h)] e^{i \xi x} d \xi \]

\[ 0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} [(A_0 + B_0) \cosh(\xi h) + A_0 \xi h \sinh(\xi h)] e^{i \xi x} d \xi . \]

Now, if

\[ v_0(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{v}_0(\xi) e^{i \xi x} d \xi \]

then,

\[ \tilde{v}_0(\xi) = \int_{-\infty}^{\infty} v_0(x) e^{i \xi x} dx \]

Using this notation and taking the inversion of (23) and (24), we obtain

\[ A_0 \cosh(\xi h) + B_0 \sinh(\xi h) = \tilde{v}_0(\xi) \]

\[ (A_0 + B_0) \cosh(\xi h) + A_0 \xi h \sinh(\xi h) = 0 . \]

Solving these equations, we get

\[ A_0 = \frac{2 \tilde{v}_0(\xi) \cosh(\xi h)}{2 \xi h - \sinh(2 \xi h)} \]

\[ B_0 = \frac{\tilde{v}_0(\xi) [xh - xh \cosh(2 \xi h) - \sinh(2 \xi h)]}{\sinh(\xi h) [2 \xi h - \sinh(2 \xi h)]} . \]

Equating terms containing \( \varepsilon \), we get

\[ 0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} [A_1 \xi h \cosh(\xi h) + B_1 \sinh(\xi h)] e^{i \xi x} d \xi \]

\[ \frac{P h}{\mu} N(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [(A_1 + B_1) \cosh(\xi h) + A_1 \xi h \sinh(\xi h)] e^{i \xi x} d \xi . \]

By Fourier Integral theorem, if we write

\[ N(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} N(\xi) e^{i \xi x} d \xi \]

then we obtain

\[ \tilde{N}(\xi) = \int_{-\infty}^{\infty} N(x) e^{i \xi x} dx. \]

Using this notation and taking the inversion of (29) and (30), we get
\[ A_1 x \cosh(xh) + B_1 \sinh(xh) = 0 \]  \( (31) \)
\[ (A_1 + B_1) \cosh(xh) + A_1 x \sinh(xh) = \frac{\Phi \tilde{N}(x)}{\mu} . \]  \( (32) \)

Solving these equations, we get

\[ A_1 = \frac{-2 \Phi \tilde{N}(x) \sinh(xh)}{\mu(2xh - \sinh(2xh))} \]  \( (33) \)
\[ B_1 = \frac{2 \Phi \tilde{N}(x) \cosh(xh)}{\mu(2xh - \sinh(2xh))} \]  \( (34) \)

Hence the values of \( u, v \) and \( p \) in (5) are given by

\[ u = \frac{p}{2\mu} \left( y^2 - h^2 \right) \]
\[ - \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{v}_0(\xi) \{ \cosh(\xi h) \cosh(\xi y) + \xi y \sinh(\xi y) \} e^{ix\xi} \sinh(\xi h) \{ 2\xi h - \sinh(2\xi h) \} d\xi \]
\[ - \frac{p \phi}{\pi \mu} \int_{-\infty}^{\infty} N(x) \frac{\xi h \{ \cosh(\xi h) \cosh(\xi y) \sinh(\xi h) \{ 2\xi h - \sinh(2\xi h) \} e^{ix\xi} }{2\xi h - \sinh(2\xi h) } d\xi \]
\[ + \frac{p \phi}{\pi \mu} \int_{-\infty}^{\infty} N(x) \frac{\xi h \{ \cosh(\xi h) \cosh(\xi y) \sinh(\xi h) \{ 2\xi h - \sinh(2\xi h) \} e^{ix\xi} }{2\xi h - \sinh(2\xi h) } d\xi \]

Then

\[ p = p_0 + \frac{2\mu \phi}{\pi} \int_{-\infty}^{\infty} \tilde{v}_0(\xi) \xi \frac{\cosh(\xi h) \cosh(\xi y) e^{ix\xi} }{2\xi h - \sinh(2\xi h) } d\xi \]
\[ - \frac{2p \phi}{\pi} \int_{-\infty}^{\infty} N(x) \frac{\xi \sinh(\xi h) \cosh(\xi y) e^{ix\xi} }{2\xi h - \sinh(2\xi h) } d\xi \]  \( (37) \)

**Sinusoidal Variation.** In this section, we shall consider the particular case of sinusoidal variation. Therefore, let

\[ e \tilde{N}(x) = e \sin \frac{2\pi x}{\lambda} \text{, where } \frac{e}{h} \ll 1 \]

Therefore

\[ \tilde{N}((x) = \pi i \left[ \delta \left( \xi + \frac{2\pi}{\lambda} \right) - \delta \left( \xi - \frac{2\pi}{\lambda} \right) \right] \]  \( (38) \)

where \( \delta \) is the Dirac’s delta function (Sneddon, 1951). Then we have

\[ u = \frac{-1}{2\pi i} \int_{-\infty}^{\infty} \tilde{v}(\xi) \left[ \sinh(2\xi h) \{ \cosh(\xi y) + \xi y \sinh(\xi y) \} + (\xi - \xi h \cosh(2\xi h) - \sinh(2\xi h) \cosh(\xi y) \sinh(\xi h) \{ 2\xi h - \sinh(2\xi h) \} e^{ix\xi} \right] d\xi \]
\[ - \frac{p \phi}{\pi \mu} \int_{-\infty}^{\infty} f(\xi) \pi i \left[ \delta \left( \xi + \frac{2\pi}{\lambda} \right) - \delta \left( \xi - \frac{2\pi}{\lambda} \right) \right] d\xi \]
\[ - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \tilde{v}(\xi) \left[ \sinh(2\xi h) \{ \cosh(\xi y) + \xi y \sinh(\xi y) \} + (\xi - \xi h \cosh(2\xi h) - \sinh(2\xi h) \cosh(\xi y) \sinh(\xi h) \{ 2\xi h - \sinh(2\xi h) \} e^{ix\xi} \right] d\xi \]
\[ - \frac{p \phi}{\pi \mu} \int_{-\infty}^{\infty} f(\xi) \pi i \left[ f \left( -\frac{2\pi}{\lambda} \right) - f \left( \frac{2\pi}{\lambda} \right) \right] \]

Where

\[ f(\xi) = \frac{[\xi h \cosh(\xi h) \cosh(\xi y) - \sinh(\xi h)(\cosh(\xi y) + \xi y \sinh(\xi y))]e^{i\xi x}}{2\xi h - \sinh(2\xi h)} \]

Thus

\[ u = \frac{p}{2\mu}(y^2 - h^2) \]

\[ - \frac{1}{2\pi i} \int_{\infty}^{-\infty} \frac{\tilde{\vartheta}(\xi)}{\sinh(\xi h) [2\xi h - \sinh(2\xi h)]} \sinh(\xi h) (2\xi h - \sinh(2\xi h)) \cosh(\xi y) e^{i\xi x} d\xi \]

\[ - \frac{2\mu e}{\mu} \frac{2\pi h}{\frac{2\pi h}{A} \cosh \frac{2\pi h}{A} \cosh \frac{2\pi y}{A} - \frac{2\pi y}{A} \sinh \frac{2\pi h}{A} \sinh \frac{2\pi y}{A} - \sinh \frac{2\pi h}{A} \cosh \frac{2\pi y}{A}}{4\pi h - \sinh \frac{4\pi h}{A}} \]

Similarly, we obtain

\[ v = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\tilde{\vartheta}(\xi)}{\sinh(\xi h) [2\xi h - \sinh(2\xi h)]} (\xi h \cosh(\xi y) + (\xi h - \xi h \cosh(2\xi h) - \sinh(2\xi h)) \sinh(\xi y)) e^{i\xi x} d\xi \]

\[ + \frac{2\mu e}{\mu} \frac{2\pi h}{\frac{2\pi h}{A} \cosh \frac{2\pi h}{A} \cosh \frac{2\pi y}{A} - \frac{2\pi y}{A} \sinh \frac{2\pi h}{A} \sinh \frac{2\pi y}{A} - \sinh \frac{2\pi h}{A} \cosh \frac{2\pi y}{A}}{4\pi h - \sinh \frac{4\pi h}{A}} \]

\[ p = p_0 + \frac{2\mu}{\pi} \frac{\tilde{\vartheta}(\xi)}{2\xi h - \sinh(2\xi h)} e^{i\xi x} d\xi - \frac{4\mu e}{\mu} \frac{2\pi h}{\frac{2\pi h}{A} \cosh \frac{2\pi h}{A} \cosh \frac{2\pi y}{A} - \cos \frac{2\pi y}{A}}{4\pi h - \sinh \frac{4\pi h}{A}} \]

REFERENCES


